THE TEMPERATURE FIELD OF A HOLLOW CYLINDER FOR SMALL FOURIER NUMBER

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Inzhenerno-Fizicheskii Zhurnal, Vol. 9. No. 5, pp. 674–679, 1965

UDC 536. 12

An approximate solution is given for the problem of the temperature distribution in an infinite hollow cylinder at small Fourier numbers.

The temperature field $u(r,\tau)$ in an infinite hollow cylinder whose inner and outer surfaces are held at the constant temperature t_1 is determined by the differential equation

$$\frac{\partial u}{\partial \tau} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad r_0 \leqslant r \leqslant R, \tag{1}$$

with the boundary and initial conditions

$$u(r_0, \tau) = t_1, \tag{2}$$

$$u(R, \tau) = t_1, \tag{3}$$

$$u(r, 0) = t_i. \tag{4}$$

The solution of the problem [1–3] is in the form of an infinite series, which converges rapidly only for sufficiently large values of the Fourier number $Fo = a\tau/r_0^2$. However, when the time from the beginning of the process is very short, the number of terms of the series required for convergence is so large that the use of this solution becomes impractical.

As has been pointed out by A. V. Lykov [2,3] the method of infinite Laplace transforms makes it possible to obtain solutions to transient heat transfer problems in different forms, and in particular in forms appropriate to small Fourier numbers. Applying the Laplace transform to (1), taking account of (4), we obtain an ordinary differential equation for the transformed temperature, whose solution is

$$\bar{u}(r, p) - t_i/p = C_1 I_0\left(\sqrt{p/a}\, r\right) + C_2 K_0\left(\sqrt{p/a}\, r\right).$$

The transformed boundary conditions are

$$\bar{u}(r_0, p) = t_1/p,$$

$$\bar{u}(R, p) = t_1/p.$$

Taking account of these boundary conditons, we obtain the transformed solution

$$\bar{u} - \frac{t_i}{p} = \frac{t_1 - t_i}{p}\left\{\left[K_0\left(\sqrt{\frac{p}{a}}R\right) - K_0\left(\sqrt{\frac{p}{a}}r_0\right)\right] \times I_0\left(\sqrt{\frac{p}{a}}r\right) + \left[I_0\left(\sqrt{\frac{p}{a}}r_0\right) - I_0\left(\sqrt{\frac{p}{a}}R\right)\right] \times K_0\left(\sqrt{\frac{p}{a}}r\right)\right\}\left[I_0\left(\sqrt{\frac{p}{a}}r_0\right)K_0\left(\sqrt{\frac{p}{a}}R\right) - I_0\left(\sqrt{\frac{p}{a}}R\right)K_0\left(\sqrt{\frac{p}{a}}r_0\right)\right]^{-1}.$$

To find a solution which would be appropriate for small Fourier numbers, we use the asymptotic expansions of the modified Bessel functions,

$$I_0(x) = \frac{\exp x}{\sqrt{2\pi x}} f(x), \quad K_0(x) = \frac{\sqrt{\pi}\exp(-x)}{\sqrt{2x}}, \tag{5}$$

where

$$f(x) = 1 + \frac{1}{8x} + \frac{9}{128x^2} + \ldots .$$

Introducing the notation $\sqrt{p/a} = q$ we obtain after some simple transformations

$$\bar{u} - \frac{t_i}{p} = \frac{t_1 - t_i}{p}\left[\frac{\operatorname{sh}(qR - qr)}{\operatorname{sh}(qR - qr_0)}\sqrt{\frac{r_0}{r}}\frac{f(qr)}{f(qr_0)} + \frac{\operatorname{sh}(qr - qr_0)}{\operatorname{sh}(qR - qr_0)}\sqrt{\frac{R}{r}}\frac{f(qr)}{f(qR)}\right].$$

Replacing sh $(qR - qr)$ and sh $(qr - qr_0)$ by their expressions in terms of exponentials, and expanding the function $1/\operatorname{sh}(qR - qr_0)$ in a series, we obtain

$$\bar{u} - \frac{t_i}{p} = \frac{t_1 - t_i}{p}\left\{\sqrt{\frac{r_0}{r}}\frac{f(qr)}{f(qr_0)} \cdot \sum_{n=1}^{\infty}\left[\exp(qR - qr - (2n-1)q(R - r_0)) - \exp(-qR + qr - (2n-1)q(R - r_0))\right] + \sqrt{\frac{R}{r}}\frac{f(qr)}{f(qR)}\sum_{n=1}^{\infty}\left[\exp(qr - qr_0 - (2n-1)q(R - r_0)) - \exp(-qr + qr_0 - (2n-1)q(R - r_0))\right]\right\}. \tag{6}$$

If we take only three terms of f(x), then, according to [2], we can write the ratios of such functions in the form

$$\frac{f(qr)}{f(qr_0)} = 1 + \frac{r_0 - r}{8qr_0 r} + \frac{9r_0^2 - 7r^2 - 2r_0 r}{128q^2 r_0^2 r^2} + \ldots,$$

$$\frac{f(qr)}{f(qR)} = 1 + \frac{R - r}{8qRr} + \frac{9R^2 - 7r^2 - 2Rr}{128q^2 R^2 r^2} + \ldots \tag{7}$$

Substituting (7) in (6) and transforming back to the original variables, we obtain the solution (in dimensionless form)

$$\Theta = \frac{u - t_1}{t_i - t_1} = 1 + \sqrt{\frac{r_0}{r}} \sum_{n=1}^{\infty} \left[\operatorname{erfc} \frac{2n(k-1) + 1 - r/r_0}{2\sqrt{\text{Fo}}} - \operatorname{erfc} \frac{2n(k-1) + 1 - 2k + r/r_0}{2\sqrt{\text{Fo}}} \right] + \sqrt{k \frac{r_0}{r}} \times$$

$$\times \sum_{n=1}^{\infty} \left[\operatorname{erfc} \frac{2n(k-1) - k + r/r_0}{2\sqrt{\text{Fo}}} - \operatorname{erfc} \frac{2n(k-1) + 2 - k - r/r_0}{2\sqrt{\text{Fo}}} \right] +$$

$$+ \frac{1}{4}\sqrt{\text{Fo}} \sqrt{\frac{r_0}{r}} \left(\frac{r_0}{r} - 1 \right) \sum_{n=1}^{\infty} \left[\operatorname{ierfc} \frac{2n(k-1) + 1 - r/r_0}{2\sqrt{\text{Fo}}} - \operatorname{ierfc} \frac{2n(k-1) + 1 - 2k + r/r_0}{2\sqrt{\text{Fo}}} \right] +$$

$$+ \frac{1}{4}\sqrt{\text{Fo}} \sqrt{\frac{r_0}{r}} \left(\frac{r_0}{r}\sqrt{k} - \sqrt{\frac{1}{k}} \right) \times \sum_{n=1}^{\infty} \left[\operatorname{ierfc} \frac{2n(k-1) - k + r/r_0}{2\sqrt{\text{Fo}}} - \operatorname{ierfc} \frac{2n(k-1) + 2 - k - r/r_0}{2\sqrt{\text{Fo}}} \right] +$$

$$+ \left[\frac{9}{32} \text{Fo} \left(\frac{r_0}{r} \right)^2 \sqrt{\frac{r_0}{r}} - \frac{7}{32} \text{Fo} \sqrt{\frac{r_0}{r}} - \frac{1}{16} \text{Fo} \left(\frac{r_0}{r} \right)^2 \sqrt{\frac{r}{r_0}} \right] \times$$

$$\times \sum_{n=1}^{\infty} \left[i^2\operatorname{erfc} \frac{2n(k-1) + 1 - r/r_0}{2\sqrt{\text{Fo}}} - i^2\operatorname{erfc} \frac{2n(k-1) + 1 - 2k + r/r_0)}{2\sqrt{\text{Fo}}} \right] +$$

$$+ \left[\frac{9}{32} \text{Fo} \left(\frac{r_0}{r} \right)^2 \sqrt{k} \sqrt{\frac{r_0}{r}} - \frac{7}{32} \text{Fo} \left(\frac{1}{k} \right)^2 \sqrt{k} \sqrt{\frac{r_0}{r}} - \frac{1}{16} \text{Fo} \left(\frac{r_0}{r} \right)^2 \sqrt{\frac{1}{k}} \sqrt{\frac{r}{r_0}} \right] \cdot$$

$$\cdot \sum_{n=1}^{\infty} \left[i^2\operatorname{erfc} \frac{2n(k-1) - k + r/r_0}{2\sqrt{\text{Fo}}} - i^2\operatorname{erfc} \frac{2n(k-1) + 2 - k - r/r_0}{2\sqrt{\text{Fo}}} \right], \tag{8}$$

where $\text{Fo} = a\tau/r_0^2$, $k = R/r_0$. Retaining only the terms containing $\operatorname{erfc} x = 1 - \operatorname{erf} x$, we obtain

$$\Theta = \frac{u - t_1}{t_i - t_1} \approx$$

$$\approx 1 + \sqrt{\frac{r_0}{r}} \sum_{n=1}^{\infty} \left[\operatorname{erf} \frac{2n(k-1) + 1 - 2k + r/r_0}{2\sqrt{\text{Fo}}} - \operatorname{erf} \frac{2n(k-1) + 1 - r/r_0}{2\sqrt{\text{Fo}}} \right] + \sqrt{k \frac{r_0}{r}} \times$$

$$\times \sum_{n=1}^{\infty} \left[\operatorname{erf} \frac{2n(k-1) + 2 - k - r/r_0}{2\sqrt{\text{Fo}}} - \operatorname{erf} \frac{2n(k-1) - k + r/r_0}{2\sqrt{\text{Fo}}} \right]. \tag{9}$$

For very small values of Fo one can use the formula

$$\Theta = \frac{u - t_1}{t_i - t_1} \approx$$

$$\approx 1 + \sqrt{\frac{r_0}{r}} \left(\operatorname{erf} \frac{r/r_0 - 1}{2\sqrt{\text{Fo}}} - \operatorname{erf} \frac{2k - 1 - r/r_0}{2\sqrt{\text{Fo}}} \right) + \sqrt{k \frac{r_0}{r}} \left(\operatorname{erf} \frac{k - r/r_0}{2\sqrt{\text{Fo}}} - \operatorname{erf} \frac{k - 2 + r/r_0}{2\sqrt{\text{Fo}}} \right). \tag{10}$$

It should be noted that one cannot get a better approximation by taking more terms in (8) and (9), as would be the case in the analogous problem for the flat plate. This is so because by using the asymptotic approximations for the modified Bessel functions (5) we have already assumed that $q = \sqrt{p/a}$ is sufficiently large. Thus, since large values of qr_0 in the transform plane correspond to small values of Fo in the original plane, it would be of no use to take more terms in (9) if Fo were not sufficiently small.

It is interesting to estimate the upper value of Fo for which equation (10) would suffice for engineering calculations. Comparing the exact solution with the solution calculated from (10), we find that these agree to four significant figures for values up to Fo = 0.01. Thus, for example, for r/r_0 = 1.25, k = 2, and Fo = 0.013 both solutions yield Θ = 0.8917. For r/r_0 = 1.75, k = 2, and Fo = 0.013 solution (10) yields Θ = 0.8707, whereas the exact value is Θ = 0.8706. Thus, for Fo $\leq$ 0.01 solution (10) agrees with the exact solution to four significant figures.

These results can be used to solve other, related, problems. For example, consider the problem of finding the temperature field in a finite hollow cylinder with all surfaces held at the constant temperature t_1. The solution to this problem $t(r, z, \tau)$ is determined by the equation

$$\frac{\partial t}{\partial \tau} = a \left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2} \right), \tag{11}$$

where

$$r_0 \leqslant r \leqslant R, \quad -l \leqslant z \leqslant l,$$

with the initial condition

$$t(r, z, 0) = t_i$$

and the boundary conditions

$$t(r_0, z, \tau) = t_1, \tag{13}$$

$$t(R, z, \tau) = t_1, \tag{14}$$

$$t(r, -l, \tau) = t_1, \tag{15}$$

$$t(r, l, \tau) = t_1. \tag{16}$$

It can be easily shown that the solution to (11)–(16) can be obtained from the solution of (1)–(4) and the solution to the problem

$$\frac{\partial v}{\partial \tau} = a \frac{\partial^2 v}{\partial z^2}, \tag{17}$$

$$v(-l, \tau) = t_1 \tag{18}$$

$$v(l, \tau) = t_1, \tag{19}$$

$$v(z, 0) = t_i. \tag{20}$$

The temperature field $t(r,z,\tau)$ is related to $u(r,\tau)$ and $v(z,\tau)$ according to

$$\frac{t - t_1}{t_i - t_1} = \frac{u - t_1}{t_i - t_1} \, \frac{v - t_1}{t_i - t_1}.$$

The solution to (17)–(20) is well known for both small and large values of $Fo_l = a\tau/l^2$ ([2], p. 85). Multiplying the solutions to (1)–(4) and (17)–(20), both taken in the form appropriate for small Fo and Fo_l, we obtain the solution to (11)–(16).

NOTATION

R and r_0) outer and inner radii of the cylinder; u, v, and t) temperature; τ) time; a) thermal diffusivity; l) half of the height of the finite cylinder; Fo and Fo_l Fourier numbers; $k = R/r_0$.

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19 July 1965

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